SURFACE WAVES AND STABILITY OF FREE SURFACE OF A MAGNETIZABLE LIQUID

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Linear theory of surface waves in a magnetizable liquid is examined. A stability criterion for a plane-free surface of a magnetizable liquid is studied for an arbitrary magnetization law.

The theory of surface waves in its classical formulation [1] leads to nonstandard, primarily nonlinear problems. Linearization of these problems is usually associated with studies of the stability of free fluid surfaces. Interest has arisen recently in this question because of the application of fluids that can be substantially magnetized or polarized in an electromagnetic field [2-5].

In this article, potential wave motions on the surface of an incompressible fluid which may be nonhomogeneously and isotropically magnetized in an applied magnetic field, are considered. We will also assume that this fluid is inviscid and does not conduct a current while its temperature remains invariant. so that the magnetization function of the medium M can be written in the form

$$\mathbf{M} = \mathbf{M}(\rho, H) = (\mathbf{H} / H)\ M(\rho, H)$$

Suppose the magnetized fluid in the undisturbed state occupies a half-space $y < 0$ while a medium is found in the region $y > 0$ whose density and magnetization may be set equal to 0.

The initial system of equations for describing surface waves consists in eddyless motion equations of an incompressible, nonconducting magnetized liquid [6]

$$\begin{gathered}\operatorname{rot} \mathbf{v} = 0, \quad \operatorname{div} \mathbf{v} = 0, \quad \frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{p + \psi^{(\rho)}}{\rho} + \frac{v^2}{2}\right) = \mathbf{g} + \frac{M}{\rho}|\nabla H \\ \operatorname{rot} \mathbf{H} = 0, \quad \operatorname{div} \mathbf{H} = -4\pi \operatorname{div}(M\mathbf{H} / H) \\ \left(\psi^{(\rho)} \equiv \int_0^H \left\{M - \rho\left(\frac{\partial M}{\partial \rho}\right)_H\right\} dH\right)\end{gathered} \tag{1}$$

and at conditions which must be satisfied on the free surface $y = F(x, y, z)$. These conditions have the form

$$F_t + \nabla F \cdot \mathbf{v}^- = v_y \tag{2}$$

$$p^- + \psi^{(\rho)} - p^+ = -2\pi(\mathbf{M}\cdot\mathbf{n})^2 + \alpha(R_1^{-1} + R_2^{-1}) \tag{3}$$

$$\mathbf{B}^-\cdot\mathbf{n} = \mathbf{H}^+\cdot\mathbf{n}, \ \mathbf{H}_\tau^- = \mathbf{H}_\tau^+ \tag{4}$$

Here, $n = \{-F_x(1 + F_x^2 + F_z^2)^{-1/2}, (1 + F_x^2 + F_z^2)^{-1/2}, -F_z(1 + F_x^2 + F_z^2)^{-1/2}\}$ is the normal basis vector to the free surface directed towards $y > F$, α is the surface tension coefficient, and R_1 and R_2 are the principal radii of curvature of the free surface,

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}(\rho, H), \quad F_t \equiv \frac{\partial F}{\partial t}, \quad F_x \equiv \frac{\partial F}{\partial x}, \quad F_z \equiv \frac{\partial F}{\partial z}$$

the superscript minuses denoting the values of the corresponding variables at the surface from the direction of the liquid, while the pluses denote these values at the surface from the nonmagnetic medium occupying the region $y > F$.

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Condition (2) denotes motion continuity of the liquid on the free surface, condition (3) asserts that the normal stresses on both sides of the free surface are equal, while condition (4) insures that the normal components of magnetic induction and the tangential components of the magnetic field are equal.

We will moreover assume that

$$\mathbf{H}_+|_{y=+\infty} = \mathbf{H}_0, \quad \mathbf{H}_-|_{y=-\infty} \equiv \mathbf{H}_\infty = (H_{0x},\ H_{0y}\mu_\infty^{-1},\ H_{0z}) \tag{5}$$

at a distance from the free surface, where H_0 is the applied field, and μ_∞ is the value of the medium magnetic permeability at a distance from the free surface.

Thus $\mathbf{H}_- - \mathbf{H}_\infty \equiv \mathbf{H}_i$ is the field induced in the magnetic due to its properties as a "liquid magnet" while $\mathbf{H}_+ - \mathbf{H}_0$ is the field induced in the space over the elementary magnet. Our problem is an example of the motion problem for a magnetized medium in which neglecting the induced field leads to known incorrect results.

We find by introducing the velocity potential $\varphi(t, x, y, z)$ ($v = \Delta\varphi$) and the magnetic field potential $\Phi(t, x, y, z)$ ($\mathbf{H} = \nabla\Phi$), using Eq. (1), equations for φ and Φ and the motion-equation integral in the form

$$\begin{gathered}\Delta\varphi = 0, \quad \Delta\Phi = -4\pi \operatorname{div}\{M\nabla\Phi / |\nabla\Phi|\} \\ \varphi_t + {}^1/_2(\varphi_x^2 + \varphi_y^2 + \varphi_z^2) + gy + (p + \psi^{(\rho)}) / \rho - (1/\rho)\int M dH = C(t)\end{gathered} \tag{6}$$

Writing the integral (6) for points of the free surface and eliminating $p + \psi^{(\rho)}$ from it by means of condition (3) we obtain a system of equations for determining the functions $F(t, x, z)$, $\varphi(t, x, y, z)$, $\Phi_+(t, x, y, z)$, and $\Phi_-(t, x, y, z)$ in the following form:

In the region $y > F(t, x, z)$

$$\Delta\Phi_+ = 0, \quad \nabla\Phi_+|_{y=+\infty} = \mathbf{H}_0 \tag{7}$$

In the region $y < F(t, x, z)$

$$\Delta\varphi = 0, \quad \nabla\varphi|_{y=-\infty} = 0 \tag{8}$$

$$\Delta\Phi_- = -4\pi \operatorname{div}\{M\nabla\Phi_- / |\nabla\Phi_-|\}, \quad \nabla\Phi_-|_{y=-\infty} = \mathbf{H}_\infty \tag{9}$$

We have the following conditions on the surface $y = F(t, x, z)$ by virtue of Eqs. (2), (4), and (6)

$$F_t = \varphi_y - F_x\varphi_x^- - F_z\varphi_z^- \tag{10}$$

$$\Phi_y^-(1 + 4\pi M^- / H^-) - \Phi_y^+ = F_x\{\Phi_x^-(1 + 4\pi M^-/H^-) - \Phi_x^+\} + F_z\{\Phi_z^-(1 + 4\pi M^- / H^-) - \Phi_z^+\} \tag{11}$$

$$\Phi_x^- - \Phi_x^+ + F_x(\Phi_y^- - \Phi_y^+) = 0, \quad \Phi_z^- - \Phi_z^+ + F_z(\Phi_y - \Phi^+_y) = 0 \tag{12}$$

$$\begin{gathered}\varphi_t^- + {}^1/_2\{(\varphi_x^-)^2 + (\varphi_y^-)^2 + (\varphi_z^-)^2\} + gF + \frac{\alpha}{\rho}(R_1^{-1} + R_2^{-1}) - \\ -\frac{1}{\rho}\int M dH - \frac{2\pi}{\rho}\left(\frac{M}{H}\right)^2(\Phi_y^- - \Phi_x^- F_x - \Phi_z^- F_z)^2(1 + F_x^2 + F_z^2)^{-1} = 0 \\ \varphi^- \equiv \varphi(t, x, F, z), \quad \Phi^\pm \equiv \Phi_\pm(t, x, F, z), \quad H^- = |\nabla\Phi^-|\end{gathered} \tag{13}$$

where the literal subscripts of φ, Φ, and F denote partial derivatives.

Our problem in this most complete form contains a nonlinearity not only in the boundary condition, but also in Eq. (9) for the magnetic field, the form of the surface on which the boundary conditions are imposed also obeying the definition. These facts cause significant mathematical difficulties.

Below, we shall consider a linear theory of surface waves in a magnetizable liquid, i.e., we will assume that the wave amplitude is sufficiently small in comparison with its length $2\pi/k$, so that the values $|kF|$, $|F_x|$, and $|F_z|$ are less than unity. We linearize Eq. (9) and the boundary conditions (10)-(13), discarding from it terms of the second order of smallness, assuming, moreover, that the values of the induced magnetic field $|\nabla\Phi_- - \mathbf{H}_\infty|$ and $|\nabla\Phi_+ - \mathbf{H}_0|$ are of the same order as $|kF|$.

If we take into account the fact that

$$\begin{gathered}4\pi M / H \approx 4\pi M_\infty / H_\infty + \mu_\infty c_\infty \mathbf{H}_\infty \cdot (\nabla\Phi_- - \mathbf{H}_\infty) \\ \mu_\infty = 1 + 4\pi M_\infty / H_\infty, \quad c_\infty = \frac{4\pi}{\mu_\infty H_\infty^2}\left\{\left(\frac{\partial M}{\partial H}\right)_\infty - \frac{M_\infty}{H_\infty}\right\}\end{gathered}$$

we find from Eqs. (7)-(13) the linear equations

$$\Delta\Phi_+ = 0, \quad \Delta\Phi_- = -c_\infty \mathbf{H}_\infty \cdot \nabla(\mathbf{H}_\infty \cdot \nabla\Phi_-), \quad \Delta\varphi = 0 \tag{14}$$

with the same conditions at infinity and with linearized conditions on the surface $y = F$

$$F_t = \varphi_y^- \tag{15}$$

$$\mu_\infty \{\Phi^-_y + c_\infty H_{0y}\, \mu_\infty^{-1} \mathbf{H}_\infty \cdot (\nabla\Phi^- - \mathbf{H}_\infty)\} - \Phi_y^+ - (\mu_\infty - 1)\ \ (F_x H_{0x} + F_z H_{0z}) = 0$$

$$\Phi_x^- - \Phi_x^+ + F_x H_{0y} (\mu_\infty^{-1} - 1) = 0, \quad \Phi_z^- - \Phi_z^+ + F_z H_{0y} (\mu_\infty^{-1} - 1) = 0 \tag{16}$$

$$\varphi_t + gF - \frac{\alpha}{\rho}(F_{xx} + F_{zz}) - \frac{(\mu_\infty - 1)^2 H_{0y}}{4\pi\rho\mu_\infty}(\Phi_y^- - H_{0x}F_x - H_{0z}F_z) - \frac{(\mu_\infty - 1)(\mu_\infty + c_\infty H_{0y}^2)}{4\pi\rho\mu_\infty}\mathbf{H}_\infty \cdot (\nabla\Phi^- - \mathbf{H}_\infty) = 0$$

We may exclude the form of the surface from the boundary conditions by means of Eq. (15) if the remaining conditions are differentiated with respect to time. We obtain, in place of Eqs. (15) and (16),

$$\mu_\infty \{\Phi_{yt}^- + c_\infty H_{0y} \mu_\infty^{-1} (\mathbf{H}_\infty \cdot \nabla\Phi_t^-)\} - \Phi_{yt}^+ - (\mu_\infty - 1)(\varphi_{yx} H_{0x} + \varphi_{yz} H_{0z}) = 0$$

$$\Phi_{xt}^- - \Phi_{xt}^+ + \varphi_{yx} H_{0y} (\mu_\infty^{-1} - 1) = 0, \quad \Phi_{zt}^- - \Phi_{zt}^+ + \varphi_{yz} H_{0y} (\mu_\infty^{-1} - 1) = 0 \tag{17}$$

$$\varphi_{tt} + g\varphi_y - \frac{\alpha}{\rho}(\varphi_{yxx} + \varphi_{yzz}) - \frac{(\mu_\infty - 1)^2 H_{0y}}{4\pi\rho\mu_\infty}(\Phi_{yt}^- - H_{0x}\varphi_{xy} - H_{0z}\varphi_{zy}) - \frac{(\mu_\infty - 1)(\mu_\infty + c_\infty H_{0y}^2)}{4\pi\rho\mu_\infty}(\mathbf{H}_\infty \cdot \nabla\Phi^-) = 0$$

Forward surface waves are described by solutions of Eq. (14) of the form

$$\begin{aligned}
\Phi_+ &= c^+ \exp\{-ky + i(\mathbf{k}\cdot\mathbf{r}) + i\omega t\} + \mathbf{H}_0\cdot\mathbf{r} \\
\Phi_- &= c^- \exp\{k'y + i(\mathbf{k}\cdot\mathbf{r}) + i\omega t\} + \mathbf{H}_\infty\cdot\mathbf{r} \\
\varphi &= (i\omega / k) \exp\{ky + i(\mathbf{k}\cdot\mathbf{r}) + i\omega t\} \\
\mathbf{k} &= (k_x, k_z), \quad k = \sqrt{k_x^2 + k_z^2} \\
\frac{k'}{k} &= \frac{\sqrt{A} - ic_\infty H_{\infty y} H_k}{1 + c_\infty H_{\infty y}^2}, \quad A \equiv 1 + c_\infty H_\infty^2, \quad H_k \equiv \frac{\mathbf{H}_0 \cdot \mathbf{k}}{k}
\end{aligned} \tag{18}$$

These functions satisfy Eqs. (14) and the boundary conditions at infinity for any c^+, c^-, and $\omega (k > 0, A > 0)$.

The constants c^+ and c^- and the phase velocity $\lambda \equiv \omega/k$ of the waves are determined from conditions (17) by substituting (18) in them, that is

$$\begin{aligned}
c^- &= (\mu_\infty - 1)\ \ (H_{\infty y} + iH_k)(1 + \mu_\infty \sqrt{A})^{-1} \\
c^+ &= c^- - H_{0y}(1 - \mu_\infty^{-1}) \\
\lambda^2 &= \lambda_0^2 + \frac{(\mu_\infty - 1)^2 (H_k^2 - H_{\infty y}^2 \sqrt{A})}{4\pi\rho(1 + \mu_\infty \sqrt{A})}
\end{aligned} \tag{19}$$

where $\lambda_0 \equiv (g/k + \alpha k/\rho)^{1/2}$ is the phase velocity in the absence of a magnetic field.

The energy-flow velocity of these forward waves is given by

$$\begin{aligned}
U &\equiv \frac{d}{dk}(\lambda k) = \frac{\lambda}{2} + \frac{1}{\lambda}\left(\frac{\alpha k}{\rho} + \frac{3}{2}\lambda_k^2 - \lambda_n^2\right) \\
\lambda_k^2 &= \frac{(\mu_\infty - 1)^2 H_k^2}{4\pi\rho(1 + \mu_\infty \sqrt{A})}, \quad \lambda_n^2 = \frac{H_{\infty y}^2 (\mu_\infty - 1)^2 \sqrt{A}}{4\pi\rho(1 + \mu_\infty \sqrt{A})}
\end{aligned} \tag{20}$$

An analysis of Eqs. (18)-(20) allows us to conclude that

1. Surface waves in a medium magnetized according to an arbitrary law are accompanied by transverse waves with decrement k'; they exist only when $\mu = \mu(H, \rho)$ since when $\mu = \mu(\rho)$ (linear magnetization) c_∞ vanishes and the decrement p' becomes real.

We have $c_\infty < 0$ for nonlinear magnetization ($\partial M/\partial H > 0$, $\partial^2 M/\partial H^2 < 0$), so that the real part of the decrement is greater in normal field and lesser in a tangential field than in a medium governed by a linear magnetization law. Consequently, the more expressed is the magnetization nonlinearity in a normal field, the narrower will be the surface layer where the disturbances are concentrated; in a tangential field, the picture is the opposite.

2. A field tangent to an undisturbed flat surface increases the phase velocity of the surface wave if $H_k = (H_0 \cdot \mathbf{k})/k \neq 0$, i.e., if this field is not perpendicular to the ave front; in this case, the energy-flow wave velocity also increases (if we neglect the influence of surface tension), so that the energy that can be transmitted by the waves through a vertical surface is greater than in a nonmagnetic medium.

3. A normal field decreases the phase velocity of the surface waves and (if we neglect surface tension) the energy-flow velocity.

4. A normal field destabilizes the free surface of a magnetized medium. When λ becomes imaginary independent of the magnitude and direction of the wave vector k, the free surface begins to become unstable. It follows from Eq. (19) that waves perpendicular to the tangential component of the applied field, i.e., when $H_k = 0$, are the most dangerous from the standpoint of the breakdown of stability of the free surface. In this case the surface can be considered stable only when

$$H_{0y}^2 < \frac{8\pi\mu_\infty \sqrt{\alpha g\rho}\,(1+\mu_\infty \sqrt{A})}{\sqrt{A}\,(\mu_\infty - 1)^2} \tag{21}$$

The de-stabilization effect of a free surface by a normal field has been confirmed by a number of experiments with ferromagnetic liquids [2, 4].

If the surface y = 0 is an interface between two media with densities ρ^+ and ρ^- isotropically magnetized according to the laws $\mu_\infty^\pm = \mu_\infty^\mp(\rho\pm, H\pm)$, an analysis similar to the one presented above leads to a stability criterion for this interface,

$$H_{0y}^2 < [8\pi\mu_\infty^+\mu_\infty^- \sqrt{\alpha g(\rho^- - \rho^+)}\,(\mu_\infty^+\sqrt{A^+} + \mu_\infty^-\sqrt{A^-})]\;[\sqrt{A^+A^-}\,(\mu_\infty^- - \mu_\infty^+)^2]^{-1}$$

$$A^\pm = 1 + c_\infty^\pm (H_\infty^\pm)^2 = 1 + \frac{4\pi}{\mu_\infty^\pm}\left\{\left(\frac{\partial M^\pm}{\partial H}\right)_{\mp\infty} - \frac{M_\infty^\pm}{H_\infty^+}\right\}$$

Here the medium with the minus superscript is below the medium with the plus superscript.

In fields H_{0y}, in which a flat free surface is unstable, there exists an equilibrium form of a free surface with a periodic structure that little differs from y = 0. Experiments with a ferromagnetic liquid [4] have demonstrated the presence of such a wavy free surface in sufficiently large normal fields.

Let us determine the form of such a free surface.

For the sake of simplicity, we will assume that we are dealing with a paramagnetic liquid ($M/H = (\mu - 1)/4\pi$ = const) and that the applied field lies in the xy plane. Then if the free surface y = F(x), which little differs from the plane y = 0 is in equilibrium, it will follow from Eq. (16) that we must have

$$\mu\Phi_y^- - \Phi_y^+ = (\mu - 1)H_{0x}, \quad \Phi_x^- - \Phi_x^+ + (\mu^{-1} - 1)F_x H_{0y} = 0$$
$$gF - \frac{\alpha}{\rho}F_{xx} - \frac{(\mu-1)^2 H_{0y}}{4\pi\rho\mu}(\Phi_y^- - H_{0x}F_x) - \frac{\mu-1}{4\pi\rho}\left(\frac{H_{0y}}{\mu}\Phi_y^- + H_{0x}\Phi_x^-\right) = 0 \tag{22}$$

where the function $\Phi\pm(x, y)$ satisfies the equations

$$\Delta\Phi_\pm = 0 \tag{23}$$

We will find the solutions of Eq. (23) in the form

$$\Phi_+ = e^{-ky}(a^+ \sin kx + b^+ \cos kx) + H_{0x}x + H_{0y}y$$
$$\Phi_- = e^{ky}(a^- \sin kx + b^+ \cos kx) + H_{0x}x + H_{0y}y\mu^{-1} \tag{24}$$

We find if we satisfy the boundary conditions (22) that

$$k = k_0 + \sqrt{k_0^2 - \rho g\alpha}, \quad a^+ = A_0a^- - B_0b^-, \quad b^+ = B_0a^- + A_0b^-$$
$$F(x) = \mu\{(\mu - 1)H_{0y}\}^{-1}\{(a^- - a^+)\sin kx + (b^- - b^+)\cos kx\}$$
$$k_0 = \frac{(\mu-1)^2(H_{0y}^2 - \mu H_{0x}^2)}{8\pi\mu\alpha(\mu+1)}, \quad A_0 = \mu\frac{\mu H_{0x}^2 - H_{0y}^2}{\mu H_{0x}^2 + H_{0y}^2}$$
$$B_0 = \frac{\mu H_{0x}H_{0y}}{\mu H_{0x}^2\ H_{0y}^2} \tag{25}$$

This solution, which satisfies all the boundary conditions for arbitrary b^-, and a^- will exist if

$$(\mu - 1)^2(H_{0y}^2 - \mu H_{0x}^2) \geqslant 8\pi\mu(\mu + 1)\sqrt{\rho g\alpha} \tag{26}$$

In this case, the phase velocity, λ, as follows from Eq. (19), becomes imaginary for any k, and consequently, the flat surface y = 0 will be unstable.

The constants a^- and b^- occurring in the function Eq. (25) remain undefined in this approximation of an unbounded liquid. If we assume that the liquid is within a sufficiently long and deep container (the length L and its depth are significantly greater than the wavelength, i.e., they are on the order of k^{-1}) and the angle of contact of its walls with the liquid is equal to θ, the constants a^- and b^- will be determined from the conditions

$$F_x|_{x=0} = -\operatorname{ctg}\theta, \quad F_x|_{x=L} = \operatorname{ctg}\theta$$

In this case, we have from Eq. (25)

$$F(x) = -k^{-1} \operatorname{ctg} \theta \, \{\sin kx + \operatorname{ctg}(kL/2) \cos kx\}$$

where $kL/2 \neq 2\pi n$ ($n = 0, \pm 1, \ldots$), since an independent resonance instability of the surface begins.

In the case of a saturated elementary magnet ($M = M_0 = \text{const}$) the criterion (26) has the form $m_0^2 (H_{0y}{}^2 - H_{0x}{}^2) \geq 16\pi \sqrt{\rho g \alpha}$, when $4\pi M_0 / H_0 = m_0 \ll 1$.

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